Exact Risk Budgeting with Return Forecasts for Asset Allocation

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Outline

- Motivation for the problem
- Problem statement

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- Prior work on the computational aspects
- Our Work
- Examples
- Summary

Motivation and History I

- Qian[2005] from PanAgora Asset Management published a white paper that introduced the concept of risk-parity for the first time.
- The idea was that in classic Markowitz portfolio optimization, equities got a very large allocation of the portfolio risk.
- Anecdotally, Bridgewater, a hedge fund from Connecticut, was marketing an 'All Weather' portfolio since 1996 that used the concept of balancing risk allocation, so this idea was not entirely new.
- Slowly the idea took more prominence as it appeared that the large equity allocation during the Lehman Brothers crisis caused many portfolios to get wiped out overnight.

Motivation and History II

- Roncalli[2010] formalized the problem of risk budgeting - a general form of risk parity which we will introduce in the next slide.
- Spinu[2013] solved risk budgeting for the first time offering an efficient solution exploiting the KKT conditions.
- Mausser & Romanko[2016] show that risk parity can be solved as a convex second order cone programming problem.
- We are extending the above work so it can used in a generic risk budgeting context.

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Given n traded assets, if $C \in \mathbb{R}^n \times \mathbb{R}^n$ is the asset variance-covariance matrix, and $x \in \mathbb{R}^n$ the vector that represents the portfolio's fractional composition, the *partial risk* of asset i under a variance risk measure can be written as $x_i(Cx)_i$.

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Then, the *risk budgeting portfolio* satisfies :

$$x_i(Cx)_i = b_i\, x^T Cx, \quad i = 1, \dots, n \tag{1}$$

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Additionally, the portfolio fully invested constraint is:

$$\sum_{i=1}^{n} x_i = 1 \tag{2}$$

Summarizing

Summarizing the above observations, we note that the risk budget problem, without return forecasts, $\mathbb{P}$ is written as:

$$\begin{aligned} \text{minimize} \quad & \sqrt{x^T C x} && (3) \\ \text{subject to} \quad & x_i(Cx)_i = b_i\, x^T C x, \quad i = 1, \ldots, n && (4) \\ & 1^T x = 1 && (5) \\ & x \geq 0 && (6) \end{aligned}$$

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Where the last condition implies no shorting.
Where b_i values are scalars will sum to a number less than equal to 1.
When all b_i values are equal to $\frac{1}{n}$, the problem is called *Risk Parity.*

The Beginning of Risk Parity I

- Qian[2005] published a white paper that introduced the concept of risk-parity for the first time
- The key idea: portfolios that invest equally in equities and bonds allocate 9 times the risk (variance) and 3 times the standard deviation
- A popular allocation like 60-40 allocates risk heavily towards equity markets

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- The key idea: portfolios that invest equally in equities and bonds allocate 9 times the risk (variance) and 3 times the standard deviation
- A popular allocation like 60-40 allocates risk heavily towards equity markets
- Risk parity is also called ERC portfolio (Equal Risk Contribution)

The Beginning of Risk Parity II

- So a equal or 60-40 allocation portfolio is not really risk balanced!
- Suggested that there is a need to come up with risk parity, where lesser of the risk goes to equity markets, which are hard to forecast and have higher variance.

The Beginning of Risk Parity II

- So a equal or 60-40 allocation portfolio is not really risk balanced!
- Suggested that there is a need to come up with risk parity, where lesser of the risk goes to equity markets, which are hard to forecast and have higher variance.
- Qian[2005] came up with the risk parity formulation, which in the vanilla case with same pairwise correlations and without transaction costs has an analytical solution
- Observations were that these portfolios seem to under-invest in equity markets, also took a small risk overall.

Literature - Economic Questions and Answers

- Roncalli[2010] discussed the idea of a **risk budget** for the first time and formalized the problem
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- An important question was addressed by Asness[2012] - why does the average investor not hold the risk parity portfolio?
- Answer - as mentioned in the previous slide, the risk parity portfolio holds lesser risk. Hence need leverage!
- Average investors do not have access to leverage.

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- Answer - as mentioned in the previous slide, the risk parity portfolio holds lesser risk. Hence need leverage!
- Average investors do not have access to leverage.
- With leverage the risk parity portfolio does indeed beat the buy and hold return profile (Asness[2012]).

Computation of Risk Budgeting Weights - Spinu I

Efficient Computation for Multi-Asset Risk Budgeting

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As shown by Spinu[2013], the risk budget problem $\mathbb{P}$ can be computed by solving an alternate problem $\mathbb{P}^*$:

$$\text{minimize} \quad \frac{1}{2}x^T Cx - \sum_{i=1}^{n} b_i \log x_i \tag{7}$$

which is a *unconstrained minimization problem* in the variable $x \in \mathbb{R}_+^n$. As a brief explanation of why this works, we can see that the first order optimality conditions for the unconstrained problem (7) read

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$$(Cx)_i - \frac{b_i}{x_i} = 0, \quad i = 1, \ldots, n,$$

which are exactly the risk budgeting conditions in $\mathbb{P}$ if we set the total variance of the portfolio in (1), without loss of generality, to 1.

Computation of Risk Budgeting Weights - Spinu II

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- It is worth-while to note that this works as equation (1) is scale-invariant - if a solution x^* satisfies (1), so does kx^*.
- Problem (7) is usually solved using some variant of Newton's method or block coordinate descent.
- Does not extend to cases where there are constraints on positions, or to when there are return forecasts that need to be incorporated.

$$x_i(Cx)_i = b_i\, x^T Cx, \quad i = 1, \ldots, n$$

Computation of Risk Parity Weights - Mausser & Romanko I

Efficient SOCP Formulation for Multi-Asset Risk Parity

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\begin{aligned}
\text{minimize} \quad & p - t && (8)\\
\text{subject to} \quad & z_i = (Cx)_i \quad i = 1, \dots, n && (9)\\
& x^T Cx \leq Np^2 && (10)\\
& x_i z_i \geq t^2 \quad i = 1, \dots, n && (11)\\
& z_i \geq 0 \quad i = 1, \dots, n && (12)\\
& p, t \geq 0 && (13)
\end{aligned}
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Here p is an upper bound of average risk and t is a lower bound on individual asset risk contributions.

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- This was a significant step in showing that risk parity or the ERC (Equal Risk Contribution) portfolio can indeed be solved very efficiently.
- Gambeta and Kwon[2022] modify this formulation to incorporate some return forecasts.
- This is still for the ERC portfolio, risk contributions are equal in this formulation.

Our Approach: Re-writing the Risk Budget Problem I

Keeping the above requirements in mind, we now consider the following mean variance with max-risk budgeting and transactions cost problem $\mathbb{P}^{**}$:

$$
\begin{aligned}
\text{minimize} \quad & -r^T x + \lambda \sqrt{x^T C x} + \mu \left\| x - x_0 \right\|_{L1} && (14) \\
\text{subject to} \quad & x_i (Cx)_i \leq b_i \, x^T C x, \quad i = 1, \ldots, n && (15) \\
& l \leq x \leq u && (16) \\
& 1^T x = 1 && (17) \\
& 1^T b = 1 && (18) \\
& b_i, x_i \geq 0 && (19)
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where μ controls transaction costs, r is used to reflect opinions on asset returns and λ is a risk-return trade-off parameter.

Our Approach: Re-writing the Risk Budget Problem II

Proposition:

If a solution to the above problem exists, the conditions on risk budget must hold with equality for all assets.

Proof by Contradiction:

Suppose there is a feasible solution to 14 and the risk budgeting constraints 15 do not hold with equality for all i, then, by definition $\exists m \in 1..n$ such that

$$x_m(Cx)_m < b_m x^T Cx \tag{20}$$

where the inequality is strict. Summing across all i, we have,

$$\sum_i x_i(Cx)_i < \sum_i b_i x^T Cx \tag{21}$$

Our Approach: Re-writing the Risk Budget Problem III

But the LHS above is the sum of all marginal risk budgets, which *must sum to the total risk.* The RHS by definition should sum to the total risk (as the partial ratios b_i sum to unity), thereby creating a contradiction. $\blacksquare$

Our Approach: Re-writing the Risk Budget Problem IV

This implies that risk-budget constraints in 14 must hold with equality if the solution exists. Rewriting the previous problem with the above result as $\mathbb{P}^{***}$:

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Therefore, 22, which is a min-risk constraint risk budgeting problem (what we define as an MRB portfolio), is an equivalent problem to 14 if we can verify that the solution satisfies the risk budget with equality.

Two Asset Case - SPY and AGG - MRB Portfolios vs. CRB Portfolios

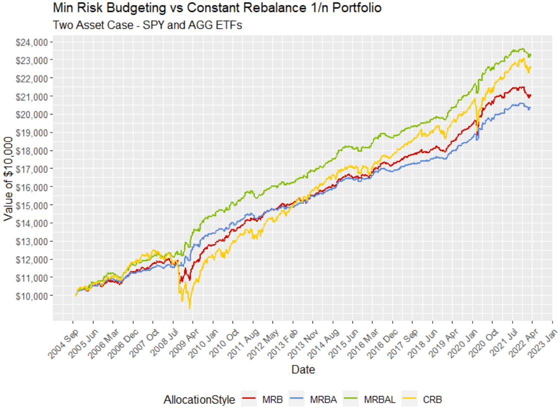

Figure 1: Backtest Results for Allocation [40%, 10%]

Four Asset Case - SPY, AGG, SPXL, GLD - MRB Portfolios vs. CRB Portfolios

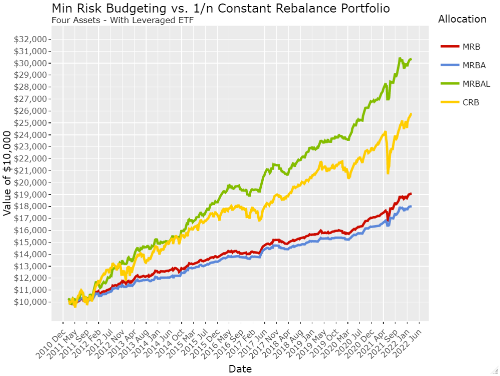

Figure 2: Backtest Results for Allocation [20%, 10%, 10%, 10%] - With Leveraged ETF

96 Assets (Nasdaq100 Index) - MRB Portfolios vs. CRB Portfolios

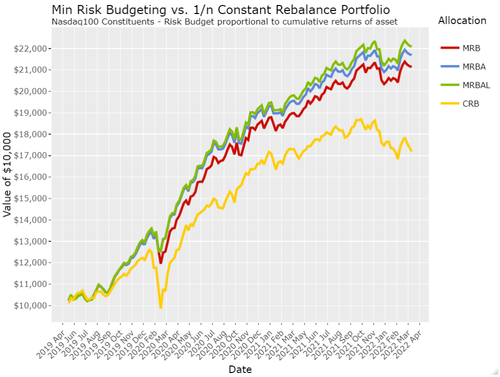

Figure 3: Backtest Results for Allocation with risk budget proportional to asset performance

Summary

- Portfolio optimization is an area where OR techniques can be used to create more stable portfolios, despite markets being very efficient.
- We have shown a powerful technique for portfolio optimization where risk and return can be traded off in a precise manner.
- This technique can be useful for sophistical retail investors and also to investment management companies.
- Further work will involve the effect of the transaction cost parameters.